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Geodesics with prescribed energy on static Lorentzian manifolds with convex boundary ★

Rossella Bartolo [a,b,1], Anna Germinario [c,*]

[a] *Departamento de Geometría y Topología, Fac. Ciencias, Univ. Granada, Avenida Fuentenueva s/n, 18071 Granada, Spain*

[b] *Dipartimento Interuniversitario di Matematica, Politecnico di Bari, Via E. Orabona, 4, 70125 Bari, Italy*

[c] *Dipartimento Interuniversitario di Matematica, Università degli Studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy*

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Abstract

By using some variational principles and the Ljusternik–Schnirelmann critical point theory, we extend some previous results dealing with the existence and multiplicity of geodesics having prescribed energy joining a point with a line on static Lorentzian manifolds with convex boundary. Our techniques work also in the case of timelike and spacelike geodesics on manifolds with boundary and for a physically relevant class of spacetimes with non-smooth boundary. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we study the existence and multiplicity of geodesics having a prescribed parameterization proportional to the arc length, joining a point with a line on a static Lorentzian manifold with convex boundary. We recall that a Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_{(L)})$ is a finite dimensional manifold $\mathcal{M}$ with a smooth, symmetric tensor field $\langle\cdot,\cdot\rangle_{(L)}$ which is a

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* Corresponding author.

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non-degenerate scalar product with index one on each $T_z\mathcal{M}$, for any $z \in \mathcal{M}$, (see e.g. [11] for more details). A *geodesic* is a curve $\gamma :]a, b[\to \mathcal{M}$ satisfying

$$D_s\dot{\gamma} = 0,$$

where D_s denotes the covariant derivative induced by the Levi-Civita connection on $\mathcal{M}$. It is easy to see that, if $\gamma :]a, b[\to \mathcal{M}$ is a geodesic, there exists a constant $E = E(\gamma) \in \mathbb{R}$ independent of s such that for any $s \in]a, b[$

$$E = \tfrac{1}{2}\langle\dot{\gamma}(s), \dot{\gamma}(s)\rangle_{(L)}.$$

In the sequel we shall call the constant E *energy*. We point out that such a name is not related to its physical meaning. A geodesic γ is said *timelike* (respectively *lightlike*, *spacelike*) if E is negative (respectively null, positive). Four-dimensional Lorentzian manifolds are the mathematical models of relativistic spacetimes. In General Relativity, lightlike geodesics represent light rays and timelike geodesics the world-lines of free falling particles.

Here we shall consider the class of static Lorentzian manifolds.

Definition 1.1. Let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ be a connected Riemannian manifold and set $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$. A *(standard) static Lorentzian metric* $\langle\cdot,\cdot\rangle_{(L)}$ on $\mathcal{M}$ is defined in the following way: for any $z = (x, t) \in \mathcal{M}$ and for any $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbb{R}$

$$\langle\zeta, \zeta'\rangle_{(L)} = \langle\xi, \xi'\rangle - \beta(x)\tau\tau', \tag{1.1}$$

where $\beta : \mathcal{M}_0 \longrightarrow \mathbb{R}$ is a smooth positive function. The couple $(\mathcal{M}, \langle\cdot,\cdot\rangle_{(L)})$ is said *static Lorentzian manifold.*

Moreover we shall assume that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is a static Lorentzian manifold with convex boundary.

Definition 1.2. Let $\mathcal{M}$ be an open subset of a manifold $\widetilde{\mathcal{M}}$ with topological boundary $\partial\mathcal{M}$. We say that $\mathcal{M}$ has *convex boundary* if for any geodesic $\gamma : [a, b] \to \mathcal{M} \cup \partial\mathcal{M}$ such that $\gamma(a), \gamma(b) \in \mathcal{M}$

$$\gamma([a, b]) \subset \mathcal{M}. \tag{1.2}$$

If $\partial\mathcal{M}$ is smooth, using the distance from the boundary, it can be proved that there exists a smooth function $\Phi : \widetilde{\mathcal{M}} \to \mathbb{R}$ such that

$$\begin{aligned} &\mathcal{M} = \{z \in \widetilde{\mathcal{M}} | \Phi(z) > 0\},\\ &\partial\mathcal{M} = \{z \in \widetilde{\mathcal{M}} | \Phi(z) = 0\},\\ &\nabla\Phi(z) \neq 0, \quad \text{for any } z \in \partial\mathcal{M}. \end{aligned} \tag{1.3}$$

Moreover it is easy to prove that if $\partial\mathcal{M}$ is smooth and convex

$$H_\Phi(z)[\zeta, \zeta] \leq 0 \tag{1.4}$$

for any $z \in \partial\mathcal{M}$ and $\zeta \in T_z\partial\mathcal{M}$, where $H_\Phi(z)$ denotes the Hessian of Φ at the point z.

For the proof of our main result we shall use the following definition.

Definition 1.3. An open subset $\mathcal{M}$ of a manifold $\widetilde{\mathcal{M}}$ with topological boundary $\partial\mathcal{M}$ is said to have *time-convex* (respectively *light-convex, space-convex*) *boundary* if (1.2) holds for any timelike (respectively lightlike, spacelike) geodesic.

Note that if $\partial\mathcal{M}$ is smooth and for instance time-convex, then (1.4) holds for any $z \in \partial\mathcal{M}$ and for any timelike vector $\zeta \in T_z\partial\mathcal{M}$.

On a static Lorentzian manifold with boundary $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ we consider a point $p = (x_0, 0) \in \mathcal{M}$ and a line $\gamma(s) = (x_1, s) \subset \mathcal{M}$. Any curve $z : [0, a] \to \mathcal{M}$, $z = (x, t)$ joining p and γ satisfies $x(0) = x_0$, $t(0) = 0$, $x(a) = x_1$. We shall denote by $\tau_{p,\gamma}(z)$ the *arrival time* given by

$$\tau_{p,\gamma}(z) = t(a).$$

Now set

$$V = -\frac{1}{\beta},$$

where β is as in Definition 1.1. The following theorem concerns the existence of geodesics of prescribed energy joining p and γ.

Theorem 1.4. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentzian manifold with smooth time-convex boundary such that $\mathcal{M}_0$ is not contractible in itself and $\mathcal{M}_0 \cup \partial\mathcal{M}_0$ is complete. Assume that there exist $L, \eta > 0$ such that*

$$\eta \le \beta(x) \le L \qquad \forall x \in \mathcal{M}_0 \tag{1.5}$$

and let $p = (x_0, 0) \in \mathcal{M}$ and $\gamma(s) = (x_1, s) \in \mathcal{M}$, $s \in \mathbb{R}$ be such that $p \notin \gamma(\mathbb{R})$. Then for any $E \in]\sup V, 0[$ there exists a sequence $(z_m)_{m\in\mathbb{N}}$ of geodesics having energy E joining p and γ such that

$$\lim_{m\to+\infty} \tau_{p,\gamma}(z_m) = +\infty. \tag{1.6}$$

Remark 1.5. *If we fix $E = 0$ (respectively $E > 0$) the previous theorem holds assuming that $\mathcal{M}$ has light-convex (respectively space-convex) boundary. Note that for the existence of a sequence of geodesics of prescribed energy joining a point with a line we need only the first inequality in (1.5); nevertheless, to avoid to state the result only for spacelike geodesics, we also assume that β is bounded from above (this assumption is satisfied by relevant physical examples of spacetimes which we shall analyze later).*

Now we consider the case when $\mathcal{M}$ has a topological boundary which is not smooth and β goes to 0 on $\partial\mathcal{M}_0$. This is the case of some physically relevant spacetimes, which will be examined in the sequel, such as the Schwarzschild and Reissner–Nordström ones, when the metric is not defined on $\partial\mathcal{M}_0$. In this situation we can reinforce the assumptions on the convexity of the boundary to control its non-smoothness. More precisely, we recall the following definition introduced in [3].

Definition 1.6. Let $\mathcal{M}$ be an open subset of a manifold $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_0 \times \mathbb{R}$, and $\partial\mathcal{M}$ its topological boundary. $\mathcal{M}$ is said to be a static Lorentzian manifold with *non-smooth convex boundary*, if

(i) $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is a static Lorentzian manifold;

(ii) $\sup_{x\in\mathcal{M}_0} \beta(x) < +\infty$;

(iii) there exists $\Phi \in C^2(\mathcal{M},]0, +\infty[)$, such that

$$\lim_{(x,t)\to z\in\partial\mathcal{M}} \Phi(x,t) = 0$$
$$\Phi(x,t) = \Phi(x,0) = \phi(x), \qquad \forall (x,t) \in \mathcal{M}$$

(iv) for every $\eta > 0$, the set $\{x \in \mathcal{M}_0 | \phi(x) \geq \eta\}$ is complete with respect to the Riemannian metric on $\mathcal{M}_0$;

(v) there exist positive constants N, M, ν and δ, such that the function Φ of (iii) satisfies the following properties:

$$\begin{aligned} &\nu \leq \langle \nabla_L \Phi(z), \nabla_L \Phi(z)\rangle_{(L)} \leq N \\ &H_\Phi(z)[\zeta,\zeta] \leq M|\langle\zeta,\zeta\rangle_{(L)}|\Phi(z) \end{aligned} \tag{1.7}$$

for any $z \in \mathcal{M}$ such that $\Phi(z) \leq \delta$ and for any $\zeta \in T_z\mathcal{M}$, where $\nabla_L\Phi$ and H_Φ denote the Lorentzian gradient and the Lorentzian Hessian of Φ, respectively.

When $\mathcal{M}$ satisfies Definition 1.6 the following theorem holds.

Theorem 1.7. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentzian manifold with non-smooth convex boundary, such that $\mathcal{M}_0$ is not contractible. Let p and γ be as in Theorem* 1.4. *Then for any $E > \sup V$ there exists a sequence of geodesics having energy E joining p and γ.*

The proofs of the above theorems will be carried out using some variational principles which allow to find geodesics with prescribed energy as critical points of suitable functionals. Indeed Lorentzian geodesics joining a point with a line on a static manifold can be found as solutions with fixed energy of a suitable Lagrangian system on $\mathcal{M}_0$ whose potential depends on β. Such solutions are critical points of a functional introduced in [14] for the study of brake orbits for a class of Hamiltonian systems. It is essentially obtained by a modified version of the classical principle of least-action. As $\mathcal{M}_0$ may be non-compact and it has a boundary, a penalization argument is needed. The penalizing term is chosen in such a way that all the critical points are solutions with energy E of some perturbed Lagrangian systems.

This kind of problem has been already studied in [4,6,7] for light rays and by using variational techniques different from ours. Strictly "negative energies" have been studied in [8] on stably causal manifolds without boundary. We point out that the presence of the boundary makes the problem more difficult, because the intrinsic approach used in [8] does not allow to handle it by the usual penalization techniques. Here we show that, at least in the static case, such techniques still work. Moreover, unlike [7], they can be applied to the study of geodesics on manifolds whose boundaries can be non-smooth and they work also in the spacelike case where the approach of [7,8] cannot be repeated. Our results have a physical interpretation when $E \le 0$. Indeed, when $E = 0$, the point p can represent a source of light and γ the world-line of an observer. In this case the lightlike geodesics joining p and γ are the images of the source seen by the observer. Due to the bending of light by gravity in some cases multiple images appear. When $E < 0$, p represents a free falling massive particle and the geodesics with fixed E joining p and γ its trajectories under the action of a gravitational field. Finally, for a physical interpretation of (1.6), see e.g. [9].

Physical examples of static Lorentzian manifolds with convex boundary are Schwarzschild and Reissner–Nordström spacetimes. Schwarzschild spacetime represents the empty spacetime outside a non-rotating, spherically symmetric massive body of radius r^*. Using the spherical coordinates $r \in]0, +\infty[$, $\theta \in]0, \pi[$, $\varphi \in]0, 2\pi[$, the metric is given by

$$ds^2 = \frac{dr^2}{\beta(r)} + r^2\, d\Omega^2 - \beta(r)\, dt^2, \tag{1.8}$$

where $d\Omega^2 = \sin^2\theta\, d\varphi^2 + d\theta^2$ is the standard metric of the unit 2-sphere S^2 in $\mathbb{R}^3$, $m > 0$ is the mass of the body and

$$\beta(r) = 1 - \frac{2m}{r}.$$

Then, (1.8) is defined on $\mathcal{M}_0 \times \mathbb{R}$ where

$$\mathcal{M}_0 = \{(r, \theta, \varphi) \in \mathbb{R}^3 | r > 2m\}.$$

Taking into account the radius of the body, there are two possible cases:

(i) $r^* > 2m$: in this case (1.8) is well defined outside the body. Indeed, if we set

$$\mathcal{M}_0^* = \{(r, \theta, \varphi) \in \mathbb{R}^3 | r > r^*\}, \tag{1.9}$$

(1.8) is defined on $\mathcal{M}_0^* \times \mathbb{R}$, which is a smooth submanifold of $\mathcal{M}_0 \times \mathbb{R}$. This is the case when the body represents a star;

(ii) $r^* < 2m$: the metric is singular on $\partial\mathcal{M}_0$ so that it is well defined on $\mathcal{M}_0 \times \mathbb{R}$. This is the case of a *black hole*: the mass of the body is so concentrate that a massive object which reaches the singularity cannot avoid its gravitational attraction.

In the case (i) $\mathcal{M}_0^* \times \mathbb{R}$ has time-convex and light-convex boundary if $r^* \in]2m, 3m[$, (see [9]). Also in the case (ii), $\mathcal{M}_0 \times \mathbb{R}$ has convex boundary according to Definition 1.6, (see [3]). Here we only recall that in (i) the function Φ is given by

$$\Phi(r) = r - r^*$$

while in (ii)

$$\Phi(r) = \sqrt{\beta(r)}.$$

Note that $\mathcal{M}^*$ is non-contractible in itself since it is homotopically equivalent to a sphere.

The Reissner–Nordström metric is the solution of the Einstein equation corresponding to the exterior gravitational field produced by a non-rotating spherically symmetric massive body electrically charged. In polar coordinates it is given by (1.8) with

$$\beta(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$$

where $e \in \mathbb{R}$ is the electric charge of the body. Note that when $e = 0$ we get the Schwarzschild metric. If $m^2 > e^2$ the equation $\beta = 0$ has two positive solutions

$$r_{\pm} = m \pm \sqrt{m^2 - e^2}.$$

Then the metric is well defined on $\mathcal{M}_0 \times \mathbb{R}$ where

$$\mathcal{M}_0 = \{(r, \theta, \varphi) \in \mathbb{R}^3 | r > r_+\}.$$

If $r^* > r_+$, defining $\mathcal{M}_0^*$ as in (1.9), it has been proved (see [9]) that $\mathcal{M}_0^* \times \mathbb{R}$ is a smooth open submanifold of $\mathcal{M}_0 \times \mathbb{R}$ having time-convex and light-convex boundary if $r^* \in]r_+, 1/2(3m + \sqrt{9m^2 - 8e^2})[$. If $r^* < r_+$, it has been proved in [3] that $\mathcal{M}_0 \times \mathbb{R}$ has convex boundary, according to Definition 1.6, provided that $m^2 > (9/5)e^2$.

Theorems 1.4 and 1.7 can be applied to the above cases to get existence and multiplicity of geodesics joining any point $p \in \mathcal{M}$ and any line γ with prescribed energy E, for any $E \in]-1, 0]$ when $r^* > r_+$ is sufficiently small and for any $E > -1$ when $r^* < r_+$ and $m^2 > (9/5)e^2$.

2. The variational framework and the penalization argument

In this section we shall state two variational principles which reduce the search of the geodesics having energy E joining p and γ to that of the critical points of a suitable functional. Before presenting them, we recall that, by the well-known Nash embedding Theorem (see [10]), the Riemannian manifold $\mathcal{M}_0$ is isometric to a submanifold of $\mathbb{R}^N$, with N sufficiently large, equipped with the metric induced by the Euclidean metric in $\mathbb{R}^N$. So in the sequel we shall assume that $\mathcal{M}_0$ is a submanifold of $\mathbb{R}^N$ and that $\langle \cdot, \cdot \rangle$ is the Euclidean metric. Following [14], we shall consider the functional

$$f(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds \int_0^1 (E - V(x)) \, ds \tag{2.1}$$

for $E \in \mathbb{R}$, $x \in \Omega^1(x_0, x_1, \mathcal{M}_0)$ where

$$\Omega^1 \equiv \Omega^1(x_0, x_1, \mathcal{M}_0) = \{x \in H^{1,2}([0, 1], \mathcal{M}_0) | x(0) = x_0, \quad x(1) = x_1\}$$

and

$$H^{1,2}([0, 1], \mathcal{M}_0) = \{x \in H^{1,2}([0, 1], \mathbb{R}^N) | x([0, 1]) \subset \mathcal{M}_0\}.$$

It is well known that Ω^1 is a Hilbert submanifold of $H^{1,2}([0,1], \mathcal{M}_0)$ whose tangent space at $x \in \Omega^1$ is given by

$$T_x\Omega^1 = \{\xi \in H^{1,2}([0,1], T\mathcal{M}_0) | \xi(s) \in T_{x(s)}\mathcal{M}_0 \ \ \xi(0) = 0 = \xi(1)\}.$$

The following proposition whose proof can be found in [1,12], allows the geodesics joining p and γ to be found by studying the solutions with prescribed energy of a Lagrangian system on $\mathcal{M}_0$. This approach is different from the one previously used for this kind of problem (see the variational principle in [2]).

Proposition 2.1. *Fix $E \in \mathbb{R}$ and let $x : [0, a] \to \mathcal{M}_0$ be a solution of*

$$\begin{aligned} &D_s\dot{x} = -\nabla V(x), \\ &x(0) = x_0, x(a) = x_1, \\ &\tfrac{1}{2}\langle \dot{x}, \dot{x} \rangle + V(x) = E, \end{aligned} \tag{2.2}$$

where $V = -1/\beta$. Then, set $z = (x, t)$, with

$$t(s) = \sqrt{2}\int_0^s \frac{1}{\beta(x(\tau))}\,\mathrm{d}\tau, \tag{2.3}$$

z is a geodesic of energy E joining p and γ. Vice-versa, consider a geodesic $z = (x, t)$ of energy E joining p and γ and such that $\beta(x)\dot{t} = \sqrt{2}$. Then x is a solution of (2.2).

Now we can state the following variational principle for (2.2). For the proof see [1,14].

Proposition 2.2. *Let $V \in C^2(\mathcal{M}_0, \mathbb{R})$ bounded from above and $E \in \mathbb{R}$, $E > \sup V$.*

(i) *Let $x \in \Omega^1$ be a critical point of f with $f(x) > 0$. Then $y(s) = x(\omega s)$, $s \in [0, \frac{1}{\omega}]$, where*

$$\omega^2(x) = \frac{\int_0^1 (E - V(x))\,\mathrm{d}s}{\frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s}, \tag{2.4}$$

is a solution of (2.2).

(ii) *Let y be a solution of* (2.2). *Then $x(s) = y(as)$, $s \in [0, 1]$ is a critical point of f on Ω^1 with $f(x) > 0$.*

Hence, in order to prove Theorems 1.4 and 1.7, we have to look for the critical points of the functional f. Let us recall the well-known Palais–Smale condition.

Definition 2.3. Let (X, h) be a Riemannian manifold modelled on a Hilbert space and let $F \in C^1(X, \mathbb{R})$. We say that F satisfies the *Palais–Smale condition* if every sequence $(x_m)_{m\in N}$ such that

$$(F(x_m))_{m\in N} \text{ is bounded} \tag{2.5}$$

and

$$\|\nabla F(x_m)\| \to 0, \quad \text{as} \quad m \longrightarrow +\infty \tag{2.6}$$

contains a converging subsequence, where $\nabla F(x)$ denotes the gradient of F at the point x with respect to the metric h and $\|\cdot\|$ is the norm on the tangent bundle induced by h. A sequence satisfying (2.5) and (2.6) is said a Palais–Smale sequence.

Since we are considering non-complete manifolds, the functional (2.1) does not satisfy the Palais–Smale condition. To avoid this problem, we need to penalize f. From now on, we shall assume that $\mathcal{M}$ is a static Lorentzian manifold with smooth time-convex boundary. As $\partial\mathcal{M}_0$ is smooth, there exists a smooth function $\phi : \widetilde{\mathcal{M}}_0 \to \mathbb{R}$ satisfying

$$\begin{aligned} &\mathcal{M}_0 = \{x \in \widetilde{\mathcal{M}}_0 | \phi(x) > 0\},\\ &\partial\mathcal{M}_0 = \{x \in \widetilde{\mathcal{M}}_0 | \phi(x) = 0\},\\ &\nabla\phi(x) \neq 0, \quad \text{for any } x \in \partial\mathcal{M}_0. \end{aligned} \tag{2.7}$$

Set for any $z = (x, t) \in \widetilde{\mathcal{M}}$

$$\Phi(z) = \Phi(x, t) = \phi(x),$$

and notice that

$$\nabla_L \Phi(z) = (\nabla\phi(x), 0),$$

where $\nabla_L \Phi(z)$ denotes the Lorentzian gradient of Φ. Then Φ satisfies (1.3). For the sake of simplicity in the sequel we shall denote with the same symbol ϕ and Φ.

For any $\epsilon \in]0, 1]$, we consider a positive increasing C^2-function $\psi_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$\begin{aligned} &\psi_\epsilon(\tau) = 0, \quad \tau \leq \tfrac{1}{\epsilon},\\ &\lim_{\tau\to+\infty} \psi_\epsilon(\tau) = +\infty, \end{aligned} \tag{2.8}$$

$\psi'_\epsilon(s) > 0$, if $s > 1/\epsilon$, $\psi_\epsilon(s) \leq \psi_{\epsilon'}(s)$, for any s, $\epsilon \leq \epsilon'$ and $\psi_\epsilon(s) \geq a_\epsilon s - b_\epsilon$ for some positive numbers a_ϵ and b_ϵ. Let for any $\epsilon \in]0, 1]$, $x \in \Omega^1$

$$\begin{aligned} f_\epsilon(x) &= f(x) + \frac{1}{2}\int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle \, \mathrm{d}s \int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x_\epsilon(s))}\right) \mathrm{d}s\\ &= \frac{1}{2}\int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle \, \mathrm{d}s \int_0^1 \left(E - V(x(s)) + \psi_\epsilon\left(\frac{1}{\phi^2(x_\epsilon(s))}\right)\right) \mathrm{d}s. \end{aligned} \tag{2.9}$$

Remark 2.4. *We point out that the previous variational principles still hold, with suitable variants, for the above penalized functionals. In particular, Proposition 2.2 holds for V replaced by*

$$V_\epsilon(x) = V(x) - \psi_\epsilon\left(\frac{1}{\phi^2(x)}\right).$$

Then, if x is a critical point of f_ϵ, defining y as in Proposition 2.2 with ω given by (2.4) (where $V = V_\epsilon$) we have for any $s \in [0, 1]$

$$E = \frac{1}{2}\langle \dot{y}(s), \dot{y}(s)\rangle + V(y(s)) - \psi_\epsilon\left(\frac{1}{\phi^2(y(s))}\right). \tag{2.10}$$

Lemma 2.5. *Let $E > \sup V$ and fix $\epsilon \in]0,1]$. Let $(x_m)_{m\in N}$ be a sequence in Ω^1 and K be a constant such that*

$$f_\epsilon(x_m) \le K, \quad \text{for any} \quad m \in N, \tag{2.11}$$

then

$$\inf\{\phi(x_m(s)) | s \in [0,1], m \in N\} > 0. \tag{2.12}$$

Proof. By (2.11) it easily follows that the sequence $(x_m)_{m\in N}$ is bounded in L^2. Moreover, (2.11) and the Hölder inequality imply that

$$\sup_{m\in N} \int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x_m(s))}\right) \mathrm{d}s < +\infty.$$

Then (2.12) is a consequence of the following lemma, see e.g. [9]. □

Lemma 2.6. *Let $(x_m)_{m\in N}$ be a sequence in Ω^1 such that*

$$\sup_{m\in N} \int_0^1 \langle \dot{x}_m, \dot{x}_m\rangle \, \mathrm{d}s < +\infty \tag{2.13}$$

and let $(s_m)_{m\in N}$ in $[0,1]$ be a sequence such that

$$\lim_{m\to+\infty} \phi(x_m(s_m)) = 0. \tag{2.14}$$

Then

$$\lim_{m\to+\infty} \int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x_m(s))}\right) \mathrm{d}s = +\infty. \tag{2.15}$$

Proposition 2.7. *Assume that $E > \sup V$. Then,*

(i) *for any $\epsilon \in]0,1]$ and for any $c \in \mathbb{R}$ the sublevel*

$$f_\epsilon^c = \{x \in \Omega^1 | f_\epsilon(x) \le c\}$$

is a complete metric subspace of Ω^1;

(ii) *for any $\epsilon \in]0,1]$, f_ϵ satisfies the Palais–Smale condition.*

Proof. Let $(x_m)_{m\in N}$ be a Cauchy sequence in f_ϵ^c, then it is a Cauchy sequence also in $H^{1,2}([0,1],\mathbb{R}^N)$ and it converges to a curve x in $H^{1,2}([0,1],\mathbb{R}^N)$. Since this convergence is also uniform, by Lemma 2.5 it results that $x \in \Omega^1$ and by the continuity of f_ϵ, we obtain the first part of the proposition. Now let $(x_m)_{m\in N}$ be a Palais–Smale sequence; in particular it results that

$$\left(\int_0^1 \langle \dot{x}_m, \dot{x}_m\rangle \, \mathrm{d}s\right)_{m\in N} \quad \text{is bounded.} \tag{2.16}$$

Then, up to a subsequence, we get the existence of a $x \in H^{1,2}([0,1],\mathbb{R}^N)$ such that

$$x_m \to x \text{ weakly in } H^{1,2}([0,1],\mathbb{R}^N). \tag{2.17}$$

Arguing as in the first part of the proof, we get that $x \in \Omega^1$. Using standard arguments, it can be proved that

$$x_m \longrightarrow x \quad \text{strongly in} \quad H^{1,2}([0,1], \mathbb{R}^N). \quad \square$$

Our next aim is to prove some a priori estimates on the critical points of the penalized functionals; in particular, we shall prove the following proposition.

Proposition 2.8. *Let $(x_\epsilon)_{\epsilon \in]0,1]}$ be a family of curves of Ω^1 such that for any $\epsilon \in]0,1]$, x_ϵ is a critical point of f_ϵ and let $K \in \mathbb{R}$ be a constant such that*

$$f_\epsilon(x_\epsilon) \le K, \quad \forall \epsilon \in]0,1]. \tag{2.18}$$

Then there exists an infinitesimal and decreasing sequence $(\epsilon_m)_{m \in N}$ of numbers in $]0,1]$ such that $(x_{\epsilon_m})_{m \in N}$ strongly converges in $H^{1,2}([0,1], \mathbb{R}^N)$ to a curve $x \in \Omega^1$ which is a critical point of f.

Remark 2.9. *It is easy to see that if $(x_\epsilon)_{\epsilon \in]0,1]}$ is a family of critical points of f_ϵ such that (2.18) holds, there exists a positive real number L_1 such that*

$$\sup_{\epsilon \in]0,1]} \int_0^1 \langle \dot{x}_\epsilon, \dot{x}_\epsilon \rangle \, \mathrm{d}s = L_1. \tag{2.19}$$

Moreover, for any $\epsilon \in]0,1]$, if $x \in \Omega^1$ is a critical point of f_ϵ, then it satisfies the following equation:

$$A_\epsilon(x) D_s \dot{x} + B_\epsilon(x) \nabla V(x) = -B_\epsilon(x) \frac{2}{\phi^3(x)} \psi_\epsilon' \left(\frac{1}{\phi^2(x)} \right) \nabla \phi(x), \tag{2.20}$$

where

$$A_\epsilon(x) = \int_0^1 \left(E - V(x) + \psi_\epsilon \left(\frac{1}{\phi^2(x)} \right) \right) \mathrm{d}s, \qquad B_\epsilon(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, \mathrm{d}s.$$

According to Proposition 2.2, see also Remark 2.4, let us set

$$\omega_\epsilon^2(x) = \frac{A_\epsilon(x)}{B_\epsilon(x)}.$$

To prove Proposition 2.8 some lemmas are needed.

Lemma 2.10. *Let $(x_\epsilon)_{\epsilon \in]0,1]}$ be a family of curves of Ω^1 such that for any $\epsilon \in]0,1]$, x_ϵ is a critical point of f_ϵ. Then, setting for any $\epsilon \in]0,1]$, $s \in [0, 1/\omega_\epsilon]$, $z_\epsilon(s) = (y_\epsilon(s), t_\epsilon(s))$, where y_ϵ is given by Proposition 2.2 and*

$$t_\epsilon(s) = \sqrt{2} \int_0^s \frac{1}{\beta(y_\epsilon(\tau))} \, \mathrm{d}\tau, \tag{2.21}$$

for any $s \in [0, 1/\omega_\epsilon]$ it results

$$\frac{1}{2} \langle \dot{z}_\epsilon(s), \dot{z}_\epsilon(s) \rangle_{(L)} = E + \psi_\epsilon \left(\frac{1}{\phi^2(y_\epsilon(s))} \right). \tag{2.22}$$

Proof. Observing that for any $s \in [0, 1]$

$$\frac{1}{2}\langle \dot{z}_\epsilon(s), \dot{z}_\epsilon(s)\rangle_{(L)} = \frac{1}{2}\langle \dot{y}_\epsilon(s), \dot{y}_\epsilon(s)\rangle + V(y_\epsilon(s))$$

proof follows from Remark 2.4 and (2.10). □

Remark 2.11. *Consider the Riemannian metric on $\mathcal{M}$ given by*

$$\langle \zeta, \zeta\rangle_R = \langle \xi, \xi\rangle + \beta(x)\tau^2$$

for any $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M}$. As H_ϕ is a bilinear form

$$H_\phi(z)[\zeta, \zeta] \le c(z)\langle \zeta, \zeta\rangle_R$$

for some positive constant $c(z)$.

Let $x_\epsilon \in \Omega^1$ be a critical point of f_ϵ and let us set, for any $\epsilon \in]0, 1]$, $s \in [0, 1]$:

$$\lambda_\epsilon(s) = \frac{2}{\phi^3(x_\epsilon(s))}\psi_\epsilon'\left(\frac{1}{\phi^2(x_\epsilon(s))}\right), \tag{2.23}$$

see (2.20). The following estimate on the family $(\lambda_\epsilon)_{\epsilon \in]0,1]}$ holds.

Lemma 2.12. *There exists $\epsilon_0 \in]0, 1]$ such that the family of functions $(\lambda_\epsilon)_{\epsilon \in]0,\epsilon_0]}$ is bounded in $L^\infty([0, 1], \mathbb{R})$.*

Proof. For any $\epsilon > 0$ we set $g_\epsilon(s) = \phi(x_\epsilon(s))$, so g_ϵ is a C^2-function on $[0, 1]$. Let s_ϵ be a minimum point for g_ϵ. Since ψ_ϵ is convex, ψ_ϵ' is non-decreasing, so it results:

$$\lambda_\epsilon(s) \le \lambda_\epsilon(s_\epsilon)$$

for any $s \in [0, 1]$, hence it is enough to prove that $(\lambda_\epsilon(s_\epsilon))_{\epsilon \in]0,1]}$ is bounded and to study the case in which

$$\inf_{\epsilon \in]0,1]} \phi(x_\epsilon(s_\epsilon)) = 0. \tag{2.24}$$

Let $z_\epsilon(s) = (y_\epsilon(s), t_\epsilon(s))$ $s \in [0, 1/\omega_\epsilon]$, as in Lemma 2.10. As $y_\epsilon(s) = x_\epsilon(\omega_\epsilon s)$, where $\omega_\epsilon^2(x_\epsilon) = A_\epsilon(x_\epsilon)/B_\epsilon(x_\epsilon)$, $\tau_\epsilon = s_\epsilon(1/\omega_\epsilon)$, is a minimum point for $h_\epsilon(s) = \phi(z_\epsilon(s)) = \phi(y_\epsilon(s))$ on $[0, 1/\omega_\epsilon]$. Now let us set, for any $\epsilon \in]0, 1]$, $s \in [0, 1/\omega_\epsilon]$:

$$\mu_\epsilon(s) = \frac{2}{\phi^3(y_\epsilon(s))}\psi_\epsilon'\left(\frac{1}{\phi^2(y_\epsilon(s))}\right). \tag{2.25}$$

It is easy to see that z_ϵ satisfies the equation

$$D_s\dot{z}_\epsilon = -\mu_\epsilon(s)\nabla_L\phi(z_\epsilon)$$

and that $\lambda_\epsilon(s_\epsilon) = \mu_\epsilon(\tau_\epsilon)$ for any $\epsilon \in]0, 1]$. Moreover by (2.24)

$$\inf_{\epsilon \in]0,1]} \phi(y_\epsilon(\tau_\epsilon)) = 0. \tag{2.26}$$

Differentiating h_ϵ twice, we get:

$$0 \le h''_\epsilon(\tau_\epsilon) = H_\phi(z_\epsilon(\tau_\epsilon))[\dot z_\epsilon(\tau_\epsilon), \dot z_\epsilon(\tau_\epsilon)] - \mu_\epsilon(\tau_\epsilon)\langle \nabla_L\phi(z_\epsilon(\tau_\epsilon)), \nabla_L\phi(z_\epsilon(\tau_\epsilon))\rangle_{(L)}. \tag{2.27}$$

By Remark 2.9, it easily follows that

$$(\|y_\epsilon\|_\infty)_{\epsilon\in]0,1]} \quad \text{is bounded.} \tag{2.28}$$

Now let for any $\epsilon \in]0, 1]$

$$\widetilde{y}_\epsilon(s) = \begin{cases} y_\epsilon(s), & \text{if } s \in \left[0, \dfrac{1}{\omega_\epsilon}\right], \\ y_\epsilon\left(\dfrac{1}{\omega_\epsilon}\right) = x_1, & \text{if } s \in \left[\dfrac{1}{\omega_\epsilon}, K_1\right], \end{cases}$$

where $K_1 = \sup_{\epsilon\in]0,1]} \frac{1}{\omega_\epsilon}$. We claim that there exists a positive constant a_1 independent of ϵ such that

$$H_\phi(z_\epsilon(\tau_\epsilon))[\dot z_\epsilon(\tau_\epsilon), \dot z_\epsilon(\tau_\epsilon)] \le a_1\langle \dot z_\epsilon(\tau_\epsilon), \dot z_\epsilon(\tau_\epsilon)\rangle_{\mathbb{R}}. \tag{2.29}$$

Indeed since

$$(\widetilde{y}_\epsilon)_{\epsilon\in]0,1]} \quad \text{is bounded} \tag{2.30}$$

in $L^\infty([0, K_1], \mathbb{R})$, from Remark 2.11, (2.29) easily follows. Moreover, since 0 is a regular value for ϕ, by (1.3) we get the existence of a positive constant a_2 such that, for ϵ sufficiently small,

$$\langle \nabla\phi(y_\epsilon(\tau_\epsilon)), \nabla\phi(y_\epsilon(\tau_\epsilon))\rangle \ge a_2. \tag{2.31}$$

By (2.27), (2.29) and (2.22)

$$\mu(\tau_\epsilon) \le a_3\left[E + \psi_\epsilon\left(\frac{1}{\phi^2(z_\epsilon(\tau_\epsilon))}\right) + \beta(y_\epsilon(\tau_\epsilon))\dot t_\epsilon^2(\tau_\epsilon)\right]$$

then, as β is bounded from below,

$$\frac{2}{\phi^3(y_\epsilon(\tau_\epsilon))}\psi'_\epsilon\left(\frac{1}{\phi^2(y_\epsilon(\tau_\epsilon))}\right) \le a_4 + a_5\psi'_\epsilon\left(\frac{1}{\phi^2(y_\epsilon(\tau_\epsilon))}\right)$$

from which we get the proof. □

Remark 2.13. *Under the assumptions of Lemma 2.12, it is easy to prove that*

$$\lim_{\epsilon\to 0}\int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x_\epsilon(s))}\right) \mathrm{d}s = 0.$$

Lemma 2.14. *Let $(x_\epsilon)_{\epsilon\in]0,1]}$ be a family of points of Ω^1 such that for any $\epsilon \in]0, 1]$, x_ϵ is a critical point of f_ϵ and (2.18) holds. Then there exist an infinitesimal and decreasing sequence $(\epsilon_m)_{m\in N}$ of numbers in* $]0, 1]$ *and a curve $x \in H^{1,2}([0, 1], \mathcal{M}_0 \cup \partial\mathcal{M}_0)$ such that*

(i) *$x(s) \in \mathcal{M}_0 \cup \partial\mathcal{M}_0$, for any $s \in [0, 1]$;*

(ii) $(x_{\epsilon_m})_{m\in N}$ *converges strongly to* x *in* $H^{1,2}([0,1],\mathbb{R}^N)$;

(iii) *there exists* $\lambda \in L^2([0,1],\mathbb{R})$, $\lambda(s) \geq 0$ *almost everywhere in* $[0,1]$, $\lambda(s) = 0$ *if* $x(s) \in \mathcal{M}_0$, *such that for any* $\xi \in T_x\Omega^1(x_0, x_1, \widetilde{\mathcal{M}}_0)$:

$$\int_0^1 (E - V(x))\,\mathrm{d}s \int_0^1 \langle \dot{x}, \dot{\xi}\rangle\,\mathrm{d}s - \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s \int_0^1 \langle \nabla V(x), \xi\rangle\,\mathrm{d}s$$
$$= \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s \int_0^1 \lambda(s)\langle \nabla\phi(x), \xi\rangle\,\mathrm{d}s. \tag{2.32}$$

Proof. By Remark 2.9, there exists an infinitesimal and decreasing sequence $(\epsilon_m)_{m\in N}$ of numbers in $]0,1]$ and a curve $x \in H^{1,2}([0,1],\mathbb{R}^N)$ such that

$$x_{\epsilon_m} \to x \quad \text{weakly in} \quad H^{1,2}([0,1],\mathbb{R}^N). \tag{2.33}$$

Since $(x_{\epsilon_m})_{m\in N}$ converges to x also uniformly and $\mathcal{M}_0 \cup \partial\mathcal{M}_0$ is complete we have (i). The same arguments used to prove the strong convergence in Proposition 2.7, allow to get (ii). By Lemma 2.12, the sequence $(\lambda_{\epsilon_m})_{m\in \mathbf{N}}$ is bounded, in particular, in $L^2([0,1],\mathbb{R})$, so there exists $\lambda \in L^2([0,1],\mathbb{R})$ such that

$$\lambda_{\epsilon_m} \to \lambda \quad \text{weakly in} \quad L^2([0,1],\mathbb{R}) \tag{2.34}$$

and it results λ non-negative almost everywhere. It is well known, see e.g. [9], that by (2.33), for any $\xi \in T_x\Omega^1(x_0, x_1, \widetilde{\mathcal{M}}_0)$, set for any $m \in N$ $\xi_m(s) = P(x_m(s))[\xi]$, where, for any $y \in \widetilde{\mathcal{M}}_0$, $P(y)$ is the projection of $\mathbb{R}^N$ on $T_y\widetilde{\mathcal{M}}_0$, it results that $(\xi_m)_{m\in N}$ has a subsequence weakly convergent to ξ in $H^{1,2}([0,1],\mathbb{R}^N)$. So, since for any $m \in N$

$$0 = f'_{\epsilon_m}(x_{\epsilon_m})[\xi_{\epsilon_m}] = A_{\epsilon_m}(x_{\epsilon_m}) \int_0^1 \langle \dot{x}_{\epsilon_m}, \dot{\xi}_{\epsilon_m}\rangle\,\mathrm{d}s - B_{\epsilon_m}(x_{\epsilon_m}) \int_0^1 \langle \nabla V(x_{\epsilon_m}), \xi_{\epsilon_m}\rangle\,\mathrm{d}s$$
$$-B_{\epsilon_m}(x_{\epsilon_m}) \int_0^1 \lambda_{\epsilon_m}(s)\langle \nabla\phi(x_{\epsilon_m}), \xi_{\epsilon_m}\rangle\,\mathrm{d}s, \tag{2.35}$$

taking the limit, we get (iii). □

Now let

$$H^{2,2}([0,1],\mathbb{R}^N) = \{x \in H^{1,2}([0,1],\mathbb{R}^N) | \dot{x} \text{ is absolutely continuous},$$
$$\ddot{x} \in L^2([0,1],\mathbb{R}^N)\},$$

and let $H^{2,2}([0,1],\widetilde{\mathcal{M}}_0)$ be the subset of $H^{2,2}([0,1],\mathbb{R}^N)$ of the curves such that $x([0,1]) \subset \widetilde{\mathcal{M}}_0$. Integrating by parts (iii) of Lemma 2.14 as $\xi \in T_x\Omega^1(x_0, x_1, \widetilde{\mathcal{M}}_0)$ is arbitrary, it results that $x \in H^{2,2}([0,1],\widetilde{\mathcal{M}}_0)$ is a weak solution of the equation

$$\int_0^1 (E - V(x))\,\mathrm{d}s D_s\dot{x} + \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s \nabla V(x)$$
$$= -\lambda(s) \int_0^1 \langle \dot{x}, \dot{x}\rangle\,\mathrm{d}s \nabla\phi(x). \tag{2.36}$$

Lemma 2.15. *Let x as in Lemma 2.14. Let $s_0 \in [0, 1]$ such that $x(s_0) \in \mathcal{M}_0$. Then there exists a closed neighborhood J of s_0 such that $\lambda(s) = 0$ for any $s \in J$.*

Proof. Since $x(s_0) \in \mathcal{M}_0$ and x is continuous, there exists a closed neighborhood J of s_0 such that $x(s) \in \mathcal{M}_0$ for any $s \in J$. As the sequence $(x_{\epsilon_m})_{m\in N}$ uniformly converges to x, by (1.3), there exists $\nu \in N$ such that

$$d = \inf\{\phi(x_{\epsilon_m}(s)) | s \in J, m \geq \nu\} > 0.$$

By (2.8) for any $m \in N$, $m \geq \nu$, such that $\epsilon_m < d^2$, we have

$$\psi_{\epsilon_m}\left(\frac{1}{\phi^2(x_{\epsilon_m}(s))}\right) = 0, \quad \psi'_{\epsilon_m}\left(\frac{1}{\phi^2(x_{\epsilon_m}(s))}\right) = 0 \qquad \forall s \in J.$$

Then, taking the weak limit as $m \to \infty$ we get $\lambda(s) = 0$, for any $s \in J$. □

Now we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. Under our assumptions, by Lemma 2.14, there exists an infinitesimal and decreasing sequence $(\epsilon_m)_{m\in N}$ of numbers in $]0, 1]$ and a curve $x \in H^{2,2}([0, 1], \widetilde{\mathcal{M}}_0)$ with support in $\mathcal{M}_0 \cup \partial\mathcal{M}_0$ such that (ii) of Lemma 2.14 holds. Moreover (2.36) holds; we shall prove that the multiplier vanishes almost everywhere. Indeed, we know that $\lambda(s) = 0$ if $x(s) \in \mathcal{M}_0$; now let $\bar{s} \in]0, 1[$ such that $x(\bar{s}) \in \partial\mathcal{M}_0$ and $\ddot{x}(\bar{s})$ exists, then $\bar{s}$ is a minimum point of $g(s) = \phi(x(s))$, so $g'(\bar{s}) = 0$, $g''(\bar{s}) \geq 0$. Arguing as in Lemma 2.10, we can consider $z = (y, t)$ with y and t as in Proposition 2.2. Letting $h(s) = \phi(z(s)) = \phi(y(s))$, we shall call $\bar{\tau}$ the minimum point corresponding to $\bar{s}$. So, differentiating twice, by (1.4), we get:

$$\begin{aligned} 0 \leq h''(\bar{\tau}) &= H_\phi(z(\bar{\tau}))[\dot{z}(\bar{\tau}), \dot{z}(\bar{\tau})] + \langle \nabla_L \phi(z(\bar{\tau})), D_s \dot{z}(\bar{\tau})\rangle_{(L)} \\ &\leq -\mu(\bar{\tau})\langle \nabla_L \phi(z(\bar{\tau})), \nabla_L \phi(z(\bar{\tau}))\rangle_{(L)}, \end{aligned} \tag{2.37}$$

where μ denotes the multiplier corresponding to z. Then by (iii) of Lemma 2.14 we obtain $\mu(s) = 0$ almost everywhere. Finally, we get $z([0, \frac{1}{\omega}]) \subset \mathcal{M}$ since z is a geodesic with energy E joining two points of $\mathcal{M}$ and $\partial\mathcal{M}$ is time-convex. □

By the results of this section we get, for suitable E, the existence of a geodesic of energy E joining p and γ.

Theorem 2.16. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentzian manifold with smooth time-convex boundary such that $\mathcal{M}_0 \cup \partial\mathcal{M}_0$ is complete. Assume that there exist $L, \eta > 0$ such that (1.5) holds and let $p = (x_0, 0) \in \mathcal{M}$ and $\gamma(s) = (x_1, s) \in \mathcal{M}$, $s \in \mathbb{R}$ be such that $p \notin \gamma(\mathbb{R})$. Then, for any $E \in]\sup V, 0[$, there exists a geodesic of energy E joining p and γ.*

Proof. Since for any $\epsilon \in]0, 1]$ f_ϵ is bounded from below, satisfies the Palais–Smale condition and its sublevels are complete, f_ϵ attains its minimum at a curve $x_\epsilon \in \Omega^1$. Moreover it is easy to see that such family satisfies (2.18) for a certain K, so by Proposition 2.8, we get the existence of a critical point of f. By Proposition 2.1 and Proposition 2.2 the proof is complete. □

3. Proof of Theorem 1.4

At first, we recall some definitions and results, see e.g. [13] and [9].

Definition 3.1. Let X be a topological space. Given a subspace A of X, the category of A in X, denoted with $\mathrm{cat}_X A$, is the minimum number of closed and contractible subsets of X covering A; if A is not covered by a finite number of such subsets of X, we set $\mathrm{cat}_X A = +\infty$.

Now we recall a *min–max* theorem due to Ljusternik and Schnirelmann.

Theorem 3.2. *Let $\mathcal{N}$ be a Riemannian manifold and F a C^1-functional on $\mathcal{N}$ satisfying the Palais–Smale condition. Let us assume that $\mathcal{N}$ is complete or that every sublevel of F in $\mathcal{N}$ is complete. Set for any $k \in N$*

$$\Gamma_k = \{A \subset \mathcal{N} | \mathrm{cat}_{\mathcal{N}} A \geq k\},$$
$$c_k = \inf_{A \in \Gamma_k} \sup_{x \in A} F(x), \tag{3.1}$$

assume that Γ_k is not empty and $c_k \in \mathbb{R}$. Then c_k is a critical value of F.

For the following theorem see [5].

Theorem 3.3. *Let $\mathcal{N}$ be a non-contractible Riemannian manifold and let x_0, x_1 be two points of $\mathcal{N}$. Then there exists a sequence $(K_m)_{m \in N}$ of compact subsets in $\Omega^1(x_0, x_1, \mathcal{N})$ such that*

$$\lim_{m \to +\infty} \mathrm{cat}_{\Omega^1} K_m = +\infty.$$

We have pointed out that the functional f does not satisfy the Palais–Smale condition, nevertheless its sublevels have finite category. Indeed, the following proposition, whose proof is essentially contained in [9], holds.

Proposition 3.4. *For any $c \in \mathbb{R}$,*

$$\mathrm{cat}_{\Omega^1} f^c < +\infty.$$

Proof of Theorem 1.4. Let us consider $(K_m)_{m \in N}$ as in Theorem 3.3. Then, if we set for any $m \in N$

$$\Gamma_m = \{A \subset \Omega^1 | \mathrm{cat}_{\Omega^1} A \geq m\},$$

we get $\Gamma_m \neq \emptyset$. By Theorem 3.2, for any $m \in N$, $\epsilon \in]0, 1]$, the values

$$c_{\epsilon,m} = \inf_{A\in\Gamma_m} \sup_{x\in A} f_\epsilon(x)$$

are well defined and are critical points of f_ϵ. Take now $\alpha \in \mathbb{R}$ and set

$$f_\alpha = \{x \in \Omega^1 | f(x) \geq \alpha\}.$$

By Proposition 3.4, there exists $m = m(\alpha) \in N$ such that, for any $A \in \Gamma_m$

$$A \cap f_\alpha \neq \emptyset.$$

Hence, for any $\epsilon \in]0, 1]$, we obtain

$$\alpha \leq c_{\epsilon,m} \leq \max f_1(K_m).$$

By Proposition 2.8, since α arbitrary, the thesis follows. Indeed we get the existence of a sequence $(x_m)_{m\in N}$ of critical points of f such that $(f(x_m))_{m\in N}$ diverges, so by using the variational principles stated in Section 2, we can consider the associated sequence $z_m = (y_m, t_m)$, with

$$\tau_{p,\gamma}(z_m) = t_m \left(\frac{1}{\omega(y_m)}\right) \geq K\sqrt{f(x_m)},$$

for a suitable constant $K > 0$, from which (1.6) follows. □

4. Proof of Theorem 1.7

Now we study the case in which $\mathcal{M}$ is a static Lorentzian manifold with non-smooth convex boundary. The variational principles stated in Section 2 still hold. Moreover, considering Φ as in Definition 1.6, we can penalize the functional f as in the smooth case. Using Lemma 2.6 and standard arguments, see also Proposition 2.7, the following proposition can be proved.

Proposition 4.1. *Assume that* $E > \sup V$. *Then,*
(i) *for any* $\epsilon \in]0, 1]$ *and for any* $c \in \mathbb{R}$ *the sublevel*

$$f_\epsilon^c = \{x \in \Omega^1 | f_\epsilon(x) \leq c\}$$

is a complete metric subspace in Ω^1;
(ii) *for any* $\epsilon \in]0, 1]$, f_ϵ *satisfies the Palais–Smale condition.*

Let us observe that, arguing as in the proof of Theorem 2.16, we get the existence of a family $(x_\epsilon)_{\epsilon\in]0,1]}$ of critical points of f_ϵ such that (2.18) holds. The following lemma holds.

Lemma 4.2. *Let* $(x_\epsilon)_{\epsilon>0}$ *be a sequence of critical points of* f_ϵ *such that* (2.18) *holds. Then, there exists a positive constant* δ_1, *independent of* ϵ, *such that*

$$\phi(x_\epsilon(s)) \geq \delta_1 \quad \text{for any} \quad s \in [0, 1].$$

Proof. Arguing by contradiction, assume that there exists a sequence $(x_{\epsilon_m})_{m\in N}$ of critical points of f_{ϵ_m}, with $(\epsilon_m)_{m\in\mathrm{N}}$ decreasing and infinitesimal sequence in]0, 1], such that

$$\min_{s\in[0,1]} \phi(x_{\epsilon_m}(s)) \to 0 \qquad \text{as} \quad m \longrightarrow +\infty. \tag{4.1}$$

Following Lemma 2.10 let us consider for any x_m, the corresponding z_m and set $h_m(s) = \phi(z_m(s))$ so that

$$\phi(z_m(\tau_m)) = \min_{s\in[0,K_1]} \phi(z_m(s)) = \min_{s\in[0,1]} \phi(x_m(s)),$$

where K_1 is as in Lemma 2.12. Eventually passing to a subsequence, let

$$\tau_0 = \lim_{m\to+\infty} \tau_m.$$

By (4.1), we get the existence of $\mu > 0$ such that for m large enough and $\tau \in [\tau_0-\mu, \tau_0+\mu]$:

$$\phi(z_m(\tau)) < \delta,$$

where δ has been introduced in Definition 1.6. Hence, by (v) of Definition 1.6, it results

$$\begin{aligned} 0 \le h_m''(\tau) &= H_\phi(z_m(\tau))[\dot z_m(\tau), \dot z_m(\tau)] + \langle \nabla_L \phi(z_m(\tau)), D_s \dot z_m(\tau)\rangle_{(L)} \\ &\le M h_m(\tau) |\langle \dot z_m(\tau), \dot z_m(\tau)\rangle_{(L)}| - 2\psi_m' \left(\frac{1}{\phi^2(z_m(\tau))}\right) \frac{\nu}{\phi^3(z_m(\tau))}. \end{aligned} \tag{4.2}$$

Arguing as in Lemma 4.1 of [3], we get the proof. □

The proof of Theorem 1.7 follows from the one of Theorem 1.4 and the previous lemma.

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